**Internet Appendix for:**
**Generalized Disappointment Aversion and Asset Prices**

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### Representation Theorem for Generalized Disappointment Aversion

We construct a functional form for risk preferences as a certainty equivalent $\mu(p)$ for lottery $p$ that solves

$$u\left(\mu(p)\right) = \sum_{x_i \in X} p(x_i)u(x_i) - \theta \sum_{x_i \leq \delta\mu(p)} p(x_i)\Big(u\left(\delta\mu(p)\right) - u(x_i)\Big) \tag{A1}$$

where $x_i$ is an outcome with probability $p(x_i)$ and $\theta$ and $\delta$ are preference parameters. The final theorem delivers a linearly homogeneous version of these preferences where

$$u(x) = \begin{cases} \frac{x^\alpha}{\alpha} & \text{for } \alpha \leq 1,\ \alpha \neq 0 \\ \log(x) & \text{for } \alpha = 0 \end{cases} \tag{A2}$$

Let $X = [x_0, x^0]$ be the set of monetary outcomes and $\mathcal{L}$ be the set of finite-support lotteries on $X$. Lotteries that assign probability one to a single $x \in X$ are denoted simply as $x$. Let $\preceq$ be a binary relation on $\mathcal{L}$ using the standard notation of $p \succ q$ means $p$ is strictly preferred to $q$, $p \succeq q$ denotes weakly preferred, and $p \sim q$ denotes indifference. The standard properties of preferences are contained in the following axiom.

Axiom 1 – Chew-Dekel Class:

*(a)* Monotonicity: *For $x, y \in X$, $x \succ y$ if and only if $x > y$,*

*(b)* Preference Relation: *$\preceq$ is complete and transitive,*

*(c)* Continuity: *For all $p \in \mathcal{L}$, the sets $\{q | q \succeq p\}$ and $\{q | p \succeq q\}$ are closed, and*

*(d)* Betweenness: *For all $p, q \in \mathcal{L}$ and $\lambda \in (0,1)$, $p \succ q$ implies $p \succ \lambda p + (1 - \lambda)q \succ q$ (respectively, $p \sim q$ implies $p \sim \lambda p + (1-\lambda)q$).*

This axiom implies that the certainty equivalent of a lottery, denoted $\mu(p)$, with the property that $\mu(p) \sim p$ exists and is a well defined function (i.e., $\mu : \mathcal{L} \to X$). Characterizing the certainty equivalent function is sufficient to characterizing preferences since monotonicity implies $p \succeq q$ if and only if $\mu(p) \geq \mu(q)$. The betweenness axiom implies that $\mu(p) \sim \lambda p + (1-\lambda)\mu(p)$. It is this property that implies preferences are linear in probabilities and, therefore, closely related to expected utility (i.e., belong to the Chew-Dekel class of preferences of Chew (1983), (1989) and Dekel (1986). See Backus, Routledge, and Zin (2005)).

LEMMA: *Preferences satisfy axiom 1 if and only if there exists*

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| *(i)* | $u : X \to [0,1]$ | continuous, increasing |
| *(ii)* | $\Delta_k : X \to X$ | continuous, increasing, with $\Delta_k(m) < \Delta_{k+1}(m)$ for all $m \in X,\ k = 1, ..., K$, |
| *(iii)* | $L_k(x,m) : [0,1]^2 \to \mathbb{R}$ | continuous, decreasing in $x$, increasing in $m$ |
| *(iv)* | $\theta_k \in (-1, \infty)$ | |

*and*

$$M(p,m) = u^{-1}\left(\sum_{x_i \in X} p(x_i)u(x_i) - \sum_{k=1}^{K} \theta_k \sum_{x_i \leq \Delta_k(m)} p(x_i) L_k\left(x_i, \Delta_k(m)\right)\right) \tag{A3}$$

*such that $\mu(p) = M(p, \mu(p))$ and $p \succeq q$ if and only if $\mu(p) \geq \mu(q)$.*

PROOF: Dekel (1986) and Chew (1989) show that preferences satisfy axiom 1 if and only if there exists a function $U$ such that $\mu(p)$ uniquely solves

$$\sum_{x_i \in X} p(x_i) U\left(x_i, \mu(p)\right) = 0 \tag{A4}$$

with $U(x,m)$ continuous, increasing in $x$, decreasing in $m$ and satisfies $U(x,m) = 0$ if $x = m$. (For exposition, consider we focus on $X = [0,1]$ with the normalization that $\mu(x_0) = 0$ and $\mu(x^0) = 1$). Converting (A4) to (A3) is simply notation and algebra. Rewrite (A4) as

$$\sum_{x_i \in X} p(x_i)u(x_i) - \sum_{x_i \in X} p(x_i)\left(U(x_i, m) + u(x_i) - u(m)\right) = u(m)$$

Let $F(x, m) = U(x, m) + u(x) - u(m)$. Note $F(x, m)$ is continuous and increasing in $x$, decreasing in $m$, and $F(x, m) = 0$ if $x = m$. Since the function is continuous, we can now partition this function (note $\Delta_k(\cdot)$ are invertible). Let

$$\begin{aligned} L_1(x, z) &= F\left(x, \Delta_k^{-1}(z)\right) I(x \leq z) \\ L_k(x, z) &= F\left(x, \Delta_k^{-1}(z)\right) I(x \leq z) - \sum_{h=1}^{k-1} L_k(x, z) \quad k = 2, ..., K \end{aligned}$$

Let $z = \Delta_k(m)$ and note $L(x, \Delta_k(m)) = 0$ at $x = \Delta_k(m)$. Finally, rescale the function by $\theta_k$. ■

Expected utility preferences require the independence axiom. Independence states that $p_1 \succ p_2$ implies $\lambda p_1 + (1-\lambda)z \succ \lambda p_2 + (1-\lambda)z$ (where $z \in X$ and $\lambda \in (0,1)$). Axiom 2, below, relaxes the independence axiom by requiring that preferences satisfy independence if the choices of $p_1$ versus $p_2$ and $\lambda p_1 + (1-\lambda)z$ versus $\lambda p_2 + (1-\lambda)z$ are "disappointment comparable." We generalize Gul by defining disappointing outcomes as being sufficiently far below the certainty equivalent. The key to our generalization is a change to the definition of disappointment.

Define the set $(p_1, p_2, z, \lambda)$ as *disappointment comparable* if:

$$\begin{aligned} &(i) \quad \sum_{x_n < \delta\mu(p_1)} p_1(x_n) = \sum_{x_n < \delta\mu(p_2)} p_2(x_n), \text{ and} \\ &(ii) \quad \text{for } i = 1, 2 \text{ and for all } x_n \in \operatorname{supp}(p_i) \\ &\qquad x_n \geq \delta\mu(p_i) \rightarrow x_n \geq \delta\mu(\lambda p_i + (1-\lambda)z) \\ &\qquad x_n \leq \delta\mu(p_i) \rightarrow x_n \leq \delta\mu(\lambda p_i + (1-\lambda)z) \end{aligned} \tag{A5}$$

Disappointment comparability first requires the the lotteries $p_1$ and $p_2$ have the same disappointment likelihood relative to the disappointment threshold of $\delta\mu(p)$. Second, comparability requires that mixing in the outcome $z$ does not cause any outcome to switch from being disappointing in $p_i$ to not disappointing in $\lambda p_i + (1-\lambda)z$, or vice versa. When these two conditions are met, preferences satisfy the independence axiom.

Axiom 2 – $\delta$-Weak Independence: *There exists a $\delta \leq 1$ such that for all $p_1, p_2 \in \mathcal{L}$, $\lambda \in (0,1)$, and $z \in X$, $p_1 \succ p_2$ implies $\lambda p_1 + (1-\lambda)z \succ \lambda p_2 + (1-\lambda)z$ if $(p_1, p_2, z, \lambda)$ is disappointment comparable.*

The parameter $\delta$ specifies how far below an outcome must be below the certainty equivalent before it is considered disappointing.

Axioms 1 and 2 are sufficient to generate an interesting class of preferences.

Theorem 1: *$\preceq$ satisfies Axioms 1 and 2 if and only if they are represented by $\mu(p)$ such that $p \succeq q$ if and only if $\mu(p) \geq \mu(q)$, where:*

$$u\left(\mu(p)\right) = \sum_{x_i \in X} p(x_i)u(x_i) - \theta \sum_{x_i \leq \delta\mu(p)} p(x_i)\Big(\ell\left(u\left(\delta\mu(p)\right)\right) - \ell\left(u(x_i)\right)\Big). \quad \text{(A6)}$$

Proof of Theorem 1: The above lemma shows that axiom 1 implies the functional form of equation (A3). In addition, $\Delta_k(m)$ and $L_k(x, \delta_k(m))$ are increasing in $x$ and decreasing in $m$, ensuring $\mu(p) = M(p, \mu(p))$ exists and is unique. Finally, $L_k(x, \Delta_k(m)) = 0$ if $x = \Delta_k(m)$. To prove the theorem, we need to establish that axiom 2 implies that $K = 1$, $\Delta_1(m) = \delta m$, $L_1(x, \delta m) = \ell(\delta m) - \ell(x)$ with $\ell(\cdot)$ as an increasing funtion. For exposition, consider $u(x) = x$ and $X = [0, 1]$, since extending to general $u(x)$ and $X$ is straightforward.

The $\delta$-Weak Independence axiom can be stated as

$$\text{if} \quad \left\{ \begin{array}{ll} (i) & \mu(p_1) \geq \mu(p_2) \\ (ii) & \sum_{x_i \leq \delta\mu(p_1)} p_1(x_i) = \sum_{x_i \leq \delta\mu(p_2)} p_2(x_i) \\ (iii) & z \in \{x | x \in \text{supp}(p_i) \text{ and } x \leq \delta\mu(p_i)\} \to z \leq \delta\mu\left(\lambda p_i + (1-\lambda)x\right) \\ (iv) & z \in \{x | x \in \text{supp}(p_i) \text{ and } x \geq \delta\mu(p_i)\} \to z \geq \delta\mu\left(\lambda p_i + (1-\lambda)x\right) \end{array} \right\}$$

$$\text{then} \qquad \mu\left(\lambda p_1 + (1-\lambda)x\right) \;\; \geq \;\; \mu\left(\lambda p_2 + (1-\lambda)x\right). \tag{A7}$$

Assume $p_1,\ p_2$, and $\lambda$ satisfy the necessary conditions of equation (A7). For notation, let $\mu_j = \mu(p_j)$ and $\hat{\mu}_j = \mu\left(\lambda p_j + (1-\lambda)x\right)$. Consider equation (A3) with one cut-off ($K = 1$).

$$\begin{aligned} \hat{\mu}_j \;\; &= \;\; \lambda \sum_{x_i \in X} p_j(x_i)x_i - \lambda\theta \sum_{x_i \leq \Delta(\hat{\mu}_j)} p_j(x_i)L(x_i, \Delta(\hat{\mu}_i)) \\ &\quad +(1-\lambda)x - (1-\lambda)\theta L(x_i, \Delta(\hat{\mu}_i))I\left(x \leq \Delta(\hat{\mu}_j)\right) \end{aligned}$$

where $I(\cdot)$ is an indicator function. For $\theta > 0$ and for arbitrary $\Delta(m)$ and $L(x, \Delta(m))$, it is easy to construct lotteries such that $\hat{\mu}_1 < \hat{\mu}_2$. Satisfying Weak Independence requires that $\Delta(m) = \delta m$ (hence $K = 1$ and there can be no other cut-offs). This implies

$$\begin{aligned} \hat{\mu}_j \;\; &= \;\; \lambda\mu_j + \lambda\theta\left(\sum_{x_i \leq \delta\mu_i} p_j(x_i)L(x_i, \delta\mu_i) - \sum_{x_i \leq \delta\hat{\mu}_j} p_j(x_i)L(x_i, \delta\hat{\mu}_j)\right). \\ &\quad +(1-\lambda)\left(x - \theta L(x, \delta\hat{\mu}_j)I(x \leq \delta\hat{\mu}_j)\right) \end{aligned}$$

By parts $(iii)$ and $(iv)$ of (A7),

$$\begin{aligned} \hat{\mu}_j \;\; &= \;\; \lambda\mu_j + \lambda\theta\left(\sum_{x_i \leq \delta\mu_j} p_j(x_i)\left(L(x_i, \delta\mu_j) - L(x_i, \delta\hat{\mu}_j)\right)\right) \\ &\quad +(1-\lambda)\left(x - \theta L(x, \delta\hat{\mu}_j)I(x \leq \delta\hat{\mu}_j\right) \end{aligned} \tag{A8}$$

Again, for arbitrary $L(x, \delta m)$ we can construct lotteries such that $\hat{\mu}_2 > \hat{\mu}_1$ (violating weak independence). It must therefore be the case that the loss function is separable in $x$ and $\delta m$. The restriction that $L(x, \delta m) = 0$ if $x = \delta m$ means that an additively separable loss function must be of the form $L(m, x) = \ell(\delta m) - \ell(x)$, where $\ell(\cdot)$ is an increasing function. Inserting this into (A8) and using, by part (*ii*) of (A7) that $\sum_{x_i \leq \mu_1} p_1(x_i) = \sum_{x_i \leq \mu_2} p_2(x_i) = \Phi$ produces

$$\hat{\mu}_j \;=\; \lambda\mu_j + \lambda\theta\Phi\left[\ell(\delta\mu_j) - \ell(\delta\hat{\mu}_j)\right] + (1-\lambda)\left(x - \theta[\ell(\delta\hat{\mu}_j) - \ell(x)]I(x \leq \hat{\mu}_j)\right)$$

In this form, if $\mu_1 > \mu_2$ then $\hat{\mu}_1 > \hat{\mu}_2$ satisfying weak independence. ■

These preferences are similar to those in equation (A1). Except, here disappointing outcomes are scaled by the increasing function $\ell(\cdot)$. These preferences allow for different degrees of risk aversion over disappointing outcomes (similar to Chew (1989) weighted utility). It also has the potential to generate preferences that are analogous to the "S-shaped" valuation function in Prospect Theory (Kahneman and Tversky (1979)). Unfortunately, unless $\ell$ is linear (proportional), the preferences are not linearly homogeneous and, therefore, ill-suited for asset pricing.

The following axiom delivers linearly homogeneity of $\mu(p)$ by requiring that outcomes above and below the disappointment threshold be treated proportionally. The axiom is stated by considering two lotteries, $p_1$ and $p_2$ that have "small tails." In particular, these lotteries have the property that when mixed with a good outcome, $x^0$ (bad outcome, $x^0$) the outcomes of $p_1$ and $p_2$ are all disappointing (all not disappointing). This implies that $\ell(\cdot)$ in equation (A6) is linear.

Axiom 3 – $\delta$-Symmetry: *Given a $\delta$ from Axiom 2, for all $p_1$, $p_2 \in \mathcal{L}$ and $\lambda \in [0,1]$ such that for all $x_n \in supp(p_i)$,* $\delta\mu(\lambda p_i + (1-\lambda)x_0) \leq x_n \leq \delta\mu(\lambda x^0 + (1-\lambda)p_i)$

$$\begin{aligned} \lambda p_1 + (1-\lambda)x_0 \;&\succeq\; \lambda p_2 + (1-\lambda)x_0 \quad \textit{iff} \\ \lambda x^0 + (1-\lambda)p_1 \;&\succeq\; \lambda x^0 + (1-\lambda)p_2 \end{aligned} \tag{A9}$$

These axioms are sufficient to state and prove our main theorem.

THEOREM 2: *$\preceq$ satisfies Axioms 1-3 if and only if they are represented by $\mu(p)$ in equation (A1) such that $p \succeq q$ if and only if $\mu(p) \geq \mu(q)$.*

PROOF OF THEOREM 2: Using axiom 3, for lotteries $p_1$ and $p_2$, let $\mu_{jb} = \mu\left(\lambda x^0 + (1-\lambda)p_j\right)$ and $\mu_{jw} = \mu\left(\lambda p_j + (1-\lambda)x_0\right)$. We calculate these certainty equivalents next, using the fact that $\delta\mu_{jw} < z < \delta\mu_{jb}$ for all $z \in \text{supp}(p_j)$.

$$\begin{aligned}\mu_{jw} &= \lambda\left[\sum_{x_i \in X} p_j(x_i)x_i - \theta \sum_{x_i \leq \delta\mu_{jw}} p_j(x_i)(\ell(\delta\mu_{jw}) - \ell(x_i))\right. \\ &\qquad \left. + (1-\lambda)\left[x_0 - \theta(\ell(\delta\mu_{jw}) - \ell(x_0))\right]\right. \\ &= \lambda \sum_{x_i \in X} p_j(x_i)x_i + (1-\lambda)\left[x_0 - \theta(\ell(\delta\mu_{jw}) - \ell(x_0))\right]\end{aligned}, \tag{A10}$$

and

$$\begin{aligned}\mu_{jb} &= \lambda x^0 + (1-\lambda)\left[\sum_{x_i \in X} p_j(x_i)x_i - \theta \sum_{x_i \leq \delta\mu_{jb}} p_j(x_i)(\ell(\delta\mu_{jb}) - \ell(x_i))\right] \\ &= \lambda x^0 + (1-\lambda)\left[\sum_{x_i \in X} p_j(x_i)x_i + \theta \sum_{x_i \in X} p_j\left(\ell(x_i)\right) - \ell(\delta\mu_{jb})\right]\end{aligned}. \tag{A11}$$

$\mu_{jw}$ depends on $p_j$ only through the first moment (the first term of equation (A10)). However, $\mu_{jb}$ depends on the first moment of $p$ as well as moments generated by $\ell(\cdot)$ (note $\sum p_j(x_i)\ell(x_i)$ in equation (A11)). Symmetry requires that $\mu_{1b} \geq \mu_{2b}$ if and only if $\mu_{1w} \geq \mu_{2w}$. The only way this condition can be satisfied for all lotteries is for the penalty function, $\ell(\cdot)$ to be affine. Given we can re-scale with the parameter $\theta$, setting $\ell(x) = x$ (or more generally, $\ell(\cdot) = u(\cdot)$) is without loss of generality. ■

Note that we stated disappointment and the $\delta-$weak independence axiom using the certainty equivalent function, $\mu(p)$. Since $\mu(p)$ exists by axiom 1, stating axioms this way is simpler and, hopefully, more intuitive. For readers who are familiar with Gul (1991), it is straightforward to express the axioms purely in terms of the preference operator, $\succeq$, and lotteries. For example, to define the amount of disappointment, one can proceed as Gul and define

$$\begin{aligned} B_\delta(p) &= \left\{q|x \in \text{supp}(q) \rightarrow (\delta^{-1}x) \succeq p\right\}, \\ W_\delta(p) &= \left\{q|x \in \text{supp}(q) \rightarrow (\delta^{-1}x) \preceq p\right\}. \end{aligned}$$

where $(\delta^{-1}x)$ is the lottery that pays $(\delta^{-1}x)$ for certain. Then define a decomposition of a lottery as $\lambda \in [0,1]$ and $q, r \in \mathcal{L}$ such that $p = (1-\lambda)q + \lambda r$ with $q \in B_\delta(p)$ and $r \in W_\delta(p)$. Since lottery $q$ has zero probability on all elements below the threshold $\delta\mu(p)$ and lottery $r$ has zero probability for outcomes above the threshold, $\lambda$ is the probability of disappointment in the lottery $p$.

# References


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